equation

$$KE^+ = 2.25\bar{u}^+ - 1.3 \quad (5)$$

Equation (5) is valid for the developing and fully developed boundary layer in smooth channel flow and over a range of Reynolds numbers.

The application of Eq. (3) to the inner layer is illustrated in Fig. 4, which is based on Clark's data. The stright line shown in the figure is Eq. (5), which is seen to fit the data reasonably well in the inner region up to the lowest value of y^+ investigated, $y^+ = 5$.

Conclusions

It is found that the linear relationship ($-\overline{uv}^+ = A\,KE^+$) is, in general, not accurate when applied to pipe and channel flows. In such flows, however, the turbulent kinetic energy varies linearly with the axial component of turbulence intensity. The linearity holds for rough and smooth ducts, for the inner and outer layers, and over a range of Reynolds numbers. The data analyzed in the present study indicate the following relationships: 1) for pipe flow, Eq. (4), with $5\times10^4 \leq Re \leq 5\times10^5$, $0.1 \leq y/d \leq 1$, $x/d \geq 27$; and 2) for channel flow, Eq. (5), with $1.23\times10^4 \leq Re_c \leq 2.59\times10^5$, $y^+ \geq 5$, $x/d \geq 40$. Equations (4) and (5) are seen to be in good agreement, and they are believed to be new.

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Technical Comments

Comment on "Some Remarks on the Beck Problem"

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THE critical buckling load for a cantilever column loaded by a tangential follower of force P is given by El Naschie[1] as

$$P_{cr} = 20.19EI/\ell^2 \quad (1)$$

where EI is the bending stiffness and ℓ is the length of the column. Equation (1) represents an instability boundary for the case in which the column structure is considered to be massless with a single concentrated mass at the tip. This result is well known and often cited in the literature[2-4] based on essentially the same dynamic stability analysis as is given in Ref. 1. However, Refs. 2-4 also clearly show that this critical tangential follower load value at which dynamic instability occurs is specific to the mass distribution assumed and cannot be considered to have applicability to columns with other mass distributions.

Thus, the particular value of critical load given by Eq. (1) cannot be considered to be a static property of a tangentially loaded column, and the closeness of the numerical coefficient 20.19 in Eq. (1) to the coefficient 20.05 obtained by Beck for dynamic instability of a tangentially loaded column of uniform mass with no concentrated tip mass must be regarded as not especially significant. In fact, as shown by the work of Pflüger quoted in Refs. 2 and 3, this coefficient varies with the ratio of tip mass to column structural mass, reaching a value near 16 when the ratio of masses is unity.

In general, the critical instability conditions for a tangentially loaded cantilever column are dynamic and cannot be found by static analysis.[2-4] The process which brings about instability is frequency coalescence followed by oscillatory divergent motion. The idealized case of a massless column with tip mass is special, since it constitutes a dynamical system with a single degree of freedom. In this special case, there is only one frequency, so that no frequency coalescence can occur. Bolotin[2] suggests that in this case, instability may be viewed as occurring through coalescence of the one finite frequency with one of the infinite frequencies of the massless column. However, although this conceptual analogy may have some tutorial value, it is far from completely satisfactory since in the ordinary case of frequency coalescence the instability is oscillatory divergent, whereas in the case of a single dynamical degree of freedom the divergence is nonoscillatory. Therefore, the addition of energy to the system by the nonconservative follower forces through cyclic motion, as described for example by Von Kármán and Biot,[5] cannot occur because at least two degrees of freedom are required.

In the case of a massless column with point mass at the tip, the characteristics of the column enter only statically, as a spring constant, whose value is simply the solution of the differential equation for a uniform column loaded by tip force F_t

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = 0 \quad (2)$$

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where y is the lateral deflection, EI is the bending stiffness, and P is the tangential follower column load at the tip.

The tip boundary conditions appropriate to the follower force are

$$\text{at} \quad x=\ell: \; EI\frac{d^2y}{dx^2}=0; \; EI\frac{d^3y}{dx^3}=F_t \tag{3}$$

and the root boundary conditions are

$$\text{at} \quad x=0: \; y=0; \; \frac{dy}{dx}=0 \tag{4}$$

The solution is

$$y=F_t/k^3EI[\cos k\ell \sin kx - \sin k\ell \cos kx + \sin k\ell - kx\cos k\ell] \tag{5}$$

where $k^2=P/EI$.

For the deflection at the tip, y_t, this becomes

$$y_t=(F_t/k^3EI)(\sin k\ell - k\ell\cos k\ell) \tag{6}$$

The effective spring constant or flexibility, α_t, which enters into the equation of motion for the tip mass as a single-degree-of-freedom system is

$$\alpha_t=y_t/F_t=(\sin k\ell - k\ell\cos k\ell)/k^3EI \tag{7}$$

This formulation as a single-degree-of-freedom system is exactly that of previous authors (e.g., Ref. 2, p. 11) and corresponds to El Naschie's analysis as well

Examination of Eq. (1) for the spring constant discloses that with increasing follower force load on the column (i.e., increasing k) the spring constant α_t decreases, becoming equal to zero when

$$\sin k\ell = k\ell\cos k\ell \tag{8}$$

The column is effectively infinitely stiff to loads at the tip of this value of k and thus the frequency of the system comprising the massless column and the tip mass theoretically approaches infinity as the condition $\tan k\ell = k\ell$ is approached. Above this value of $k\ell$ the spring constant α_t is negative, representing a nonoscillatory divergent type of instability.

The spring constant for a transverse force F_a applied to a cantilever column at a point other than its tip ($x=a<\ell$), with a tangential follower load at the tip, is easily obtained by noting that, between the point of application of the transverse force and the tip, integration of Eq. (2) shows that the column has no curvature. Thus the slopes at the point of application of the transverse load and at the tip are identical and the deflection at the point of application of the transverse force is given by

$$y_a=(F_a/k^3EI)(\sin ka - ka\cos ka) \tag{9}$$

The spring constant at point a is then

$$\alpha_a=(1/k^3EI)(\sin ka - ka\cos ka) \tag{10}$$

Clearly for a massless column with a concentrated mass at a only, the value of k for which α_a vanishes varies with a, which unequivocally shows that the instability is dynamic, being dependent on the mass distribution.

To emphasize the extent to which the results obtained for the idealized massless structure with tip mass under tangential follower force loadings are special and nonrepresentative of the behavior found by analysis of more realistic models of columns under tangential follower force loading, Bolotin[2] has presented the results for a massless column with two concentrated masses at different locations, in which case there are two natural frequencies which tend to coalesce with increasing follower force load leading to oscillatory divergent instability. In a similar vein, Panovko and Gubanova[4] have shown that if the rotational inertia of the tip mass is taken into account, the resulting two frequencies also tend to coalesce with increasing follower force load and give rise to oscillatory divergent instability.

Finally, it may be noted (as El Naschie has in an earlier paper cited in Ref. 1) that Eq. (1) gives precisely the classical Euler buckling load (i.e., with conservative compression forces) of the column with one end fixed and the other pinned.[6] This follows from the fact that Eqs. (7) and (8) correspond to the condition that $y=0$ at $x=\ell$ regardless of the applied force F_t. Thus, since the other boundary conditions, Eqs. (3) and (4), are those of a fixed-pinned column, the condition $y_t=0$ leads to the classical Euler buckling load. However, it would be erroneous to assume from this fact that there is any physical similarity between the classical buckling of a fixed-pinned column and the instability of a cantilever column under tangential follower forces. The difference is obvious if Eq. (6), which indicates diminishing flexibility to transverse tip loading of a column under tangential follower forces as the latter force is increased toward the critical load, is compared with the well-known increasing flexibility to transverse loading of the classical (conservatively loaded) column as the compressive force is increased toward the classical buckling load.

Thus, although in the special case of an ideal massless structure with a single concentrated mass under tangential follower forces, the onset of instability is governed by the reversal of sign of the stiffness against transverse loads and is hence predictable from static criteria, the nature of the instability is essentially dynamic and there is no analogy to the classical Euler buckling problem. Several evidences of the dynamic nature of the problem have been indicated in the foregoing discussion, including the sensitivity of the instability load to the location of the single concentrated mass, to the addition of other concentrated masses or to the inclusion in the analysis of the distributed mass of the column structure itself.

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THE author is particularly glad that his small Note has triggered Flax's valuable comments and, in particular, his

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